# A Fast implementation of quasi-Newton LMS algorithm using FFT

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***Abstract*—In this paper, a new efficient adaptive filtering algorithm belonging to the Quasi-Newton (QN) family is proposed. In the new algorithm, the autocorrelation matrix is assumed to be Toeplitz. Due to this assumption, the algorithm can be implemented in the frequency domain using the fast Fourier transform (FFT). The proposed algorithm turns out to be particularly suitable for adaptive channel equalization in wireless burst transmission systems. The algorithm exhibits a faster convergence rate and less computational complexity, as compared with other Newton-type algorithms. The performance of the proposed algorithm is compared to that of the QN-LMS algorithm in noise cancellation and channel equalization settings.**

## I. Introduction

Adaptive filtering has been an active research area over the few decades due to its wide applicability and robustness in many signal processing and communications applications. Designing and adaptive filter requires several important factors, such as rate of convergence, computational complexity, tracking abilities and accuracy of steady-state solution, to be taken into account [1], [2]. In general, in real-time applications, the computational complexity and rate of convergence play a critical role and much efforts have been directed toward deriving adaptive filtering algorithms with relatively low computational cost and fast convergence rate without sacrificing performance.

Among the efficient algorithms of particular interest are those that could be implemented in the frequency domain [1], [3]. In fact, due to their computational efficiency and their good convergence properties, frequency domain adaptive filtering algorithms tend to perform well in many situations, [4]. It is interesting to note that very little effort has been spent towards developing frequency domain implementations of Quasi-Newton (QN) algorithms.

The QN family of algorithms lies between the least mean square (LMS) and recursive least squares (RLS) algorithms. In other words, the QN algorithms usually exhibit faster convergence rate than the LMS and lower complexity than the RLS algorithms. The main difficulty in deriving frequency domain adaptive algorithms of the QN type stems from the fact that the standard form of the inverse autocorrelation matrix does not allow convolutional operations in the resulting updating recursions.

In this paper, we propose a new approximate inverse quasi-Newton (AIQN) algorithm that replaces the inverse of the input-signal autocorrelation matrix by an approximate one, provided that the input-signal autocorrelation matrix is Toeplitz. This assumption allows replacing the update of the inverse autocorrelation matrix by the update of the autocorrelation matrix itself, and performing the multiplication of $\mathbf{R}^{-1}\mathbf{x}$ in the update equation by using the Fourier transform. It is expected that this will increase the rate of convergence of the algorithm, in one hand, and decrease its computational complexity, on the other. The number of computations needed in the update equation at one iteration become much less than those needed in the QN-LMS algorithm. The performance of the proposed algorithm is compared to that of the QN-LMS algorithm in different settings.

This paper is organized as follows. In section II, an overview of the Newton-LMS adaptive filter is presented. In section III the proposed Quasi-Newton LMS algorithm is derived. In section IV, the implementation of the approximate inversion technique and in section V, simulation results are provided. Finally, the conclusions are drawn.

## II. Newton LMS Algorithm

In this section, an overview of the Newton-LMS adaptive filter is given. Consider the Wiener-Hopf equation at time-step $k$,

$$\mathbf{R}(k)\mathbf{w}(k) = \mathbf{p}(k) \tag{1}$$

where $\mathbf{R}(k)$ is the instantaneous estimate of the autocorrelation matrix, $\mathbf{w}(k)$ is the filter weights vector and $\mathbf{p}(k)$ is the instantaneous estimate of the cross-correlation between the desired signal and the input vector. The correlations are estimated recursively as

$$\mathbf{R}(k) = \beta\mathbf{R}(k-1) + \mathbf{x}(k)\mathbf{x}^T(k) \tag{2}$$

$$\mathbf{p}(k) = \beta\mathbf{p}(k-1) + d(k)\mathbf{x}(k) \tag{3}$$

where $\beta < 1$ is the forgetting factor and is usually chosen very close to unity. Substituting (2) and (3) in (1) and using the matrix inversion lemma [5], after simplification, it gives the update equation of the Newton-LMS algorithm [2]:

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$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mu(k)\mathbf{R}^{-1}(k-1)\mathbf{x}(k)e(k) \tag{4}$$

where $e(k)$ is the a-priori estimation error given by (5) and $\mu(k)$ is the variable step-size which may be required to stabilize the recursion in the update equation. The value of $\mu(k)$ is given in (6)

$$e(k) = d(k) - \mathbf{x}^T(k)\mathbf{w}(k-1) \tag{5}$$

$$\mu(k) = \frac{1}{\beta + \mathbf{x}^T(k)\mathbf{R}^{-1}(k-1)\mathbf{x}(k)} \tag{6}$$

## III. A New Quasi-Newton LMS Algorithm

Newton-LMS in its original form is computationally very complex as it requires inversion of the autocorrelation matrix at every time step. We propose to replace the inverse by an approximate one, obtained from the Toeplitz approximation of $\mathbf{R}(k-1)$. Assuming that the approximate inverse satisfies

$$\mathbf{P}(k-1)\mathbf{R}(k-1) = \mathbf{U}(k-1) \tag{7}$$

where $\mathbf{U}(k-1)$ is a matrix having eigenvalues in the unit disc when the eigenvalue spread of $\mathbf{R}(k-1)$ is less than some certain value. The weight update equation of the proposed algorithm would be

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mu(k)\mathbf{P}(k-1)\mathbf{x}(k)e(k) \tag{8}$$

which, by making use of (7), can be written as

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mu(k)\mathbf{U}(k-1)\mathbf{R}^{-1}(k-1)\mathbf{x}(k)e(k) \tag{9}$$

Following a similar procedure for the original Newton-LMS we get

$$\begin{aligned}\bar{\mathbf{w}}(k) &= \left[\mathbf{I} - \mu(k)\alpha^{-1}\mathbf{U}(k-1)\right]\bar{\mathbf{w}}(k-1) \\ &+ \mu(k)\alpha^{-1}\mathbf{U}(k-1)\mathbf{w}_{opt}\end{aligned} \tag{10}$$

where $\mathbf{w}_{opt}$ is the optimum solution and hence, for convergence, it is sufficient that $0 < \mu < \frac{1}{\lambda_{max}(\mathbf{U}(k-1))}$

## IV. Implementation of the Approximate Inversion Technique

In this section, implementing the multiplication of $\mathbf{R}^{-1}(k-1)\mathbf{x}(k)$, or equivalently $\mathbf{P}(k-1)\mathbf{x}(k)$, using the DFT method is described in detail. The main idea is to obtain an approximate inversion of $\mathbf{R}(k-1)$ and apply transform techniques to carry out the multiplication $\mathbf{P}(k-1)\mathbf{x}(k)$ in the update equation.

Given a Toeplitz autocorrelation matrix ($\mathbf{R}$) corresponding to the autocorrelation sequence (ACS) of a stationary process:

$$r(n) = E\left\{x(k)x^*(k+n)\right\}, \quad n = -(N-1), \ldots, (N-1) \tag{11}$$

where $r(-n) = r^*(n)$. The matrix $\mathbf{R}$ can be written as:

$$\begin{bmatrix} r(0) & r(-1) & \ldots & r(-(N-1)) \\ r(1) & r(0) & \ldots & r(-(N-2)) \\ \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & \ldots & r(0) \end{bmatrix} \tag{12}$$

The power spectrum of the signal corresponding to the truncated ACS is,

$$S(\omega) = \sum_{n=-(N-1)}^{N-1} r(n)e^{-jn\omega}. \tag{13}$$

Where the inverse of $\mathbf{R}$ can be approximated using the inverse power spectrum defined as:

$$Q(\omega) = \frac{1}{S(\omega)} = \sum_{n=-(N-1)}^{N-1} q(n)e^{-jn\omega}. \tag{14}$$

Where $\mathbf{P} = Toep\{q(n)\}$ generated by the sequence $\{q(n);\ n = -(N-1), \ldots, (N-1)\}$.

Now, considering the sequence $\{q(n);\ n = -(N-1), \ldots, (N-1)\}$, the symmetric sequence $g_q(k)$ can be constructed as:

$$g_q(k) = \begin{cases} q_i, & 0 \le i \le (N-1) \\ q_i^*, & -(N-1) \le i < 0 \end{cases} \tag{15}$$

where $q_i^* = q_i$.
The $n^{th}$ element of the vector $\mathbf{p}_f(k) = \mathbf{P}(k)\mathbf{x}(k)$ can be written as:

$$p_{f,n}(k) = \sum_{m=1}^{N} q_{n,m}x_{m-1}(k), \quad n = 1, 2, \ldots, N. \tag{16}$$

Rewriting (16) in terms of the sequence in (15) gives

$$p_{f,n}(k) = \sum_{m=0}^{N-1} g_{n-m-1}x_m(k), \quad n = 1, 2, \ldots, N, \tag{17}$$

Equation (17) represents the convolution sum. Now, taking $(2N-1)$-point DFT of both sides of (17) at time $k$

$$P_{fe}(l) = G(l)X_e(l), \quad l = 1, 2, \ldots, 2N-1, \tag{18}$$

where $P_{fe}(l)$ is the DFT of the zero-padded sequence $\{p_{fe,n}(k); \quad n = 1, 2, \ldots, 2N-1\}$:

$$p_{fe,n}(k) = \begin{cases} p_{f,n}(k), & n = 1, 2, \ldots, N \\ 0, & n = N+1, \ldots, 2N-1, \end{cases} \tag{19}$$

and $X_e(l)$ is the DFT of $\mathbf{x}_e(k) = [\mathbf{x}(k)\ \underline{\mathbf{0}}]$ where $\underline{\mathbf{0}}$ is an $(N-1)$-dimensional zero vector. The sequence $\{p_{f,n}(k); \quad n = 1, 2, \ldots, N\}$ can now be recovered from the inverse DFT of $P_{fe}(l)$.

By applying this method, the computational complexity of the QN-LMS algorithm will be significantly reduced as shown in Table I.

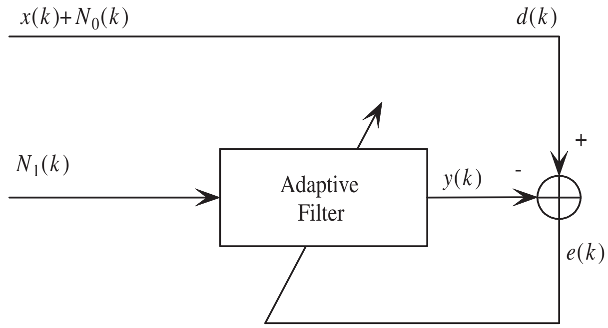


Fig. 1. Block diagram of the adaptive noise cancellation model.

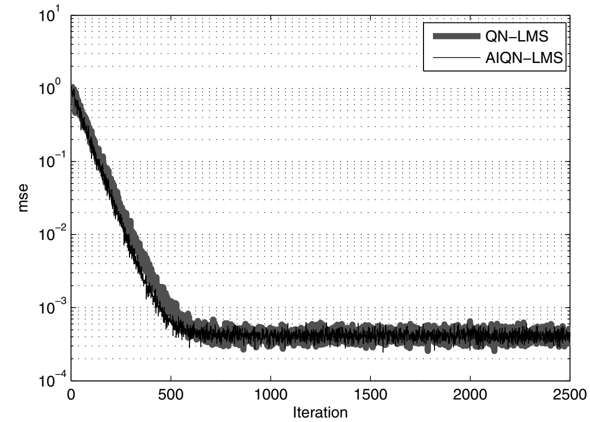


Fig. 2. Ensemble mse for AIQN-LMS and QN-LMS algorithms in AWGN for noise cancellation setting.

## V. Simulation Results

The performance of the algorithm has been compared with that of the QN-LMS algorithm in noise cancellation and channel equalization settings under additive white Gaussian noise (AWGN).

TABLE I
COMPUTATIONAL COMPLEXITY OF AIQN-LMS AND QN-LMS ALGORITHMS

| | AIQN-LMS | QN-LMS |
|---|---|---|
| Mul./Div. | $0.5N^2 + N\left[5 + 3log_2(N)\right]$ | $6N^2 + 3N + 2$ |
| Add./Sub. | $0.5N^2 + N\left[1.5 + 9log_2(N)\right]$ | $5N^2 + 3N + 3$ |

### A. Adaptive Noise Cancellation

A block diagram of the noise cancellation setting used is shown in Fig. 1. The input signal is assumed to be white Gaussian with zero mean and unity variance ($x(k) \sim G(0,1)$). The noise model is assumed to be AWGN with zero mean and variance ($\sigma_v^2 = 4 \times 10^{-4}$).

Both algorithms were implemented with the parameters: filter length of $N = 12$ taps and $\beta = 0.992$. Fig. 2 shows that both algorithms converge to the same mean-square-error (mse) (mse=-37dB). However, the proposed AIQN-LMS algorithm converges faster than the QN-LMS algorithm (AIQN-LMS converges after 520 iterations where the QN-LMS converges after 750 iterations) with a significant reduction in the computational complexity of the proposed algorithm. Table I shows the computational complexity of both algorithms.

### B. Adaptive Channel Equalization

The performance of the proposed algorithm becomes more prominent if the eigenvalue spread of the autocorrelation matrix is relatively low. To show this, we show the performance in the channel equalization setting described in [1]. The block diagram of the channel equalization model, used in this experiment, is depicted in Fig. 3. The two random-number generators (1 and 2) in the model are used to generate the transmitted signal $x_n$ and the additive noise at the receiver input, respectively. The sequence $x_n$ is a Bernoulli sequence

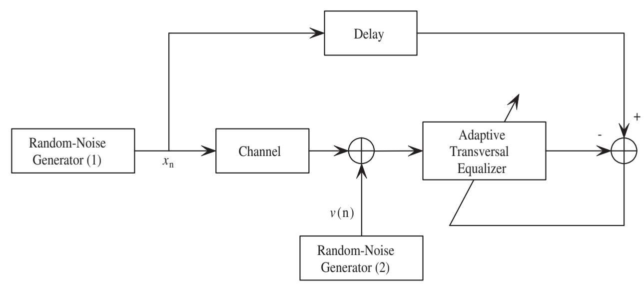


Fig. 3. Block diagram of the adaptive equalization model.

with $x_n = \pm 1$; the random variable $x_n$ has a zero mean and variance 1, and $v(n)$ is a white Gaussian with zero mean and variance dependent on the desired SNR. The impulse response of the channel is defined by: [1]

$$h(n) = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{W}(n-2)\right)\right], & n = 1,2,3, \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

where $W$ controls the eigenvalue spread of the autocorrelation matrix.

The performance of the proposed AIQN-LMS algorithm is again compared to that of the QN-LMS [1] algorithm. The algorithms were implemented with the parameters: filter length of $N = 11$ taps, signal-to-noise ratio (SNR) = 30dB, $W = 2.9$ or equivalently the eigenvalue spread of the autocorrelation matrix ($\chi(\mathbf{R}) = 6.0782$), and a delay of $\Delta = 7$. For the proposed AIQN and QN-LMS $\beta$ was selected to be unity. Fig. 4 shows that all the algorithms converge to the same mean-square-error (mse) (mse=-28dB). However, the proposed AIQN algorithm converges faster than the QN-LMS algorithm (AIQN-LMS converges after 120 iterations where the QN-LMS converges after 250 iterations) with reduced computational complexity.

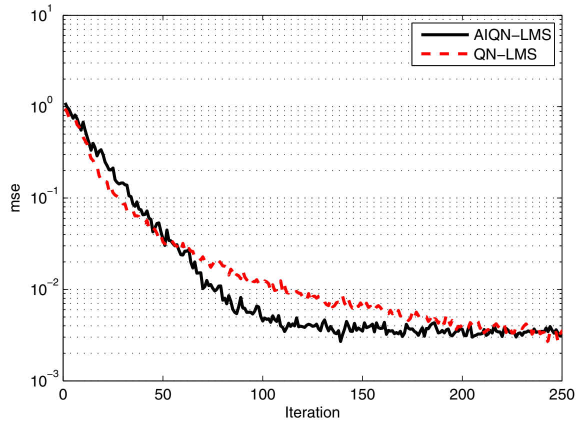


Fig. 4. Ensemble mse for AIQN-LMS and QN-LMS algorithms in AWGN for channel equalization setting.

## VI. Conclusions

In this paper, a new quasi-Newton LMS algorithm is proposed. The proposed AIQN-LMS algorithm replaces the inverse of the input-signal autocorrelation matrix by an approximate one, assuming that the input-signal autocorrelation matrix is Toeplitz. The algorithm takes advantage of the FFT which in turn leads to a significant reduction in the computational complexity without sacrificing performance. The performance of the proposed AIQN-LMS algorithm is compared to that of the QN-LMS algorithm in noise cancellation and channel equalization settings. The proposed AIQN-LMS algorithm has lower computational complexity than that of the QN-LMS algorithms with higher and/or the same performance in terms of mse and convergence rate.

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