ESTIMATION OF THE FREQUENCY AND WAVEFORM OF A SINGLE-TONE SINUSOID USING AN OFFLINE-OPTIMIZED ADAPTIVE FILTER

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ABSTRACT

A novel offline-optimized adaptive filtering (OOAF) algorithm is proposed which allows the coupled estimation of waveform and frequency of a single-tone sinusoid in additive Gaussian noise. In a specified range of frequencies, the coefficients of a FIR filter are computed to minimize a given noise power. At a particular frequency in the range, quadratic polynomials are used to compute the impulse response and its derivative with respect to frequency. The waveform and its frequency are then determined by an adaptive algorithm that uses the offline-optimized filter. In AWGN, the frequency estimate is shown to be unbiased and has an approximate asymptotic variance derived analytically. Simulation and analytical results show that OOAF exhibits excellent performance in tracking the sinusoidal waveform and estimation of its frequency especially in colored noise.

1. INTRODUCTION

Estimation of sinusoidal signals and their frequencies from noisy measurements is important in many fields such as angle of arrival estimation, frequency-shift keying (FSK) demodulation and Doppler estimation of radar waveforms [1]. The observed signal has the following general form

$$x(k) = a\cos(k\theta + \phi) + q(k),$$

where a is the amplitude, ϕ is the phase of the signal and $q(k)$ is a zero mean Gaussian noise process with variance equal to σ_q^2. The problem is to estimate the frequency θ and the waveform of the sinusoid, $\hat{x}(k)$, from the noisy observations of $x(k)$. Frequency estimation can be achieved by means of either parametric or nonparametric methods [1]. Many approaches, such as the maximum likelihood estimation and notch filtering techniques [2] exist for the estimation of frequency. Recently, adaptive algorithms have received much attention because of their high performance in waveform tracking and noise suppression [3]. In this paper, an adaptive algorithm is introduced that uses offline-optimized filters to suppress the noise and result in the estimation of the waveform and the frequency of the sinusoid. The main novelty in this work is the use of an offline-optimized filter which offers much flexibility compared to the adaptive notch filter [2] in shaping the frequency response for improved colored noise rejection. The approach resembles that of [3] which employs online constraints for waveform estimation in the frequency domain. It is well known that the Least Mean Square (LMS) algorithm does not perform well in cases where the noise is colored. In this paper, we first design the offline filter to suppress frequency components outside a specified range. We then present the structure of the adaptive filter used for tracking the frequency and the waveform of the input signal. The frequency estimate is shown to be unbiased and the approximate asymptotic variance of the proposed OOAF is given in section 4. Simulation results and conclusions are included in the end.

2. THE OFFLINE OPTIMIZATION OF FIR FILTER

Consider a predictive FIR filter of length N,

$$\hat{x}_o(k+1) = \sum_{n=0}^{N-1} h_{o,n} x_o(k-n). \tag{1}$$

In order that the filter can predict a sinusoidal signal $x_o(k) = a\cos(k\theta_1)$ at frequency θ_1, the following equations must be satisfied:

$$\sum_{n=0}^{N-1} h_{o,n}\cos(n\theta_1) = \cos\theta_1 \tag{2}$$

$$\sum_{n=0}^{N-1} h_{o,n}\sin(n\theta_1) = -\sin\theta_1. \tag{3}$$

Equations (2) and (3) can be written in matrix form as

$$\mathbf{A}\mathbf{h_o} = \mathbf{p} \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & \cos\theta_1 & \cos 2\theta_1 & \dots & \cos\{(N-1)\theta_1\} \\ 0 & \sin\theta_1 & \sin 2\theta_1 & \dots & \sin\{(N-1)\theta_1\} \end{bmatrix},$$

$$\mathbf{p} = [\cos\theta_1 \quad -\sin\theta_1]^T.$$

This filter can be optimized by minimizing a suitably chosen cost function in terms of the noise suppression performance, subject to the constraints in (4). The frequency response of the predictive filter in (1) is

$$H_o(\theta) = \sum_{n=0}^{N-1} h_{o,n} e^{-j(n+1)\theta}. \quad (5)$$

When the frequency to be estimated is in the lower part of the frequency range, the magnitude frequency response has a low-pass character. It will be assumed in the rest of this paper that θ_1 lies in a certain frequency range $[\theta_{1\min}, \theta_{1\max}]$. In order to suppress frequency components outside this range and design the frequency response of the resulting filter, the following cost function is defined,

$$J_1 = \sum_{m=1}^{M} w_m \left|H_o(\theta(m))\right|^2. \quad (6)$$

where $\theta(m),\ m = 1,\ldots,M$ is the frequency range represented by M samples, and $w_m,\ m = 1,\ldots,M$ are the weights that can be used to shape the frequency response. For equal w_m, J_1 is approximately the white noise gain (norm) of the filter. Minimization of J_1 subject to the constraints in (4), can be achieved by using the method of Lagrange multipliers. Incorporating the constraint equation (4) in the cost function, we obtain

$$J_2 = \sum_{m=1}^{M} w_m \left|H_o(\theta(m))\right|^2 + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{h_o} - \mathbf{p}). \quad (7)$$

where $\boldsymbol{\lambda} = [\lambda_1 \quad \lambda_2]^T$ is the vector of Lagrange multipliers. The minimization of J_2 with respect to $\mathbf{h_o}$ results in the following equation:

$$\sum_{i=0}^{N-1} h_{o,i} c_{i,j} = -\frac{1}{2}\left(\lambda_1 a_{1,n+1} + \lambda_2 a_{2,n+1}\right) \ n = 0,\ldots,N-1 \quad (8)$$

where

$$c_{i,j} = \sum_{m=1}^{M} w_m \cos[(j-i)\theta(m)] \quad i,j = 1,\ldots,N.$$

In (8), $a_{i,n+1},\ \ i = 1,2$ are the elements of $\mathbf{A}$ and $c_{i,j}$ are the elements of $\mathbf{C}$. Equation (8) can be written in matrix form as

$$\mathbf{C}\mathbf{h_o} = -\frac{\mathbf{1}}{\mathbf{2}}\mathbf{A}^T\boldsymbol{\lambda}. \quad (9)$$

The solution of the filter coefficient vector $\mathbf{h_0}$ can be obtained from (9) as

$$\mathbf{h_o} = \mathbf{C^{-1}}\mathbf{A}^T\left(\mathbf{A}\mathbf{C^{-1}}\mathbf{A}^T\right)^{\mathbf{-1}}\mathbf{p}. \quad (10)$$

Now, $\mathbf{h_o}$ can be calculated as a function of θ_1 in the range of frequencies $[\theta_{1\min}, \theta_{1\max}]$.

3. THE ADAPTIVE FILTER

Consider an adaptive predictive FIR filter of the form

$$\hat{x}(k+1) = \sum_{n=0}^{N-1} h_n(k)x(k-n) = \mathbf{h}^T(k)\mathbf{x}_N(k) \quad (11)$$

where $\hat{x}(k+1)$ is the output of the filter.

$$\mathbf{h}(k) = [h_0(k)\ldots h_{N-1}(k)]^T \quad (12)$$

is the vector of filter coefficients at time step k and

$$\mathbf{x}_N(k) = [x(k)\ldots x(k-N+1)]^T \quad (13)$$

is the observed data vector where $x(k) = x_o(k)+q(k)$. The prediction error is defined as

$$e_p(k+1) = x(k+1) - \hat{x}(k+1). \quad (14)$$

Given an initial coefficient vector $\mathbf{h}(k) = \mathbf{h}_o[\hat{\theta}_1(k)]$, where $\hat{\theta}_1(k)$ is the estimate of the frequency at step k, the problem is to adjust $\mathbf{h}$ so that the filter output matches the original signal x_o. At step k,

$$\hat{x}(k+1) = \mathbf{h}^T(k)\mathbf{x}_N(k). \quad (15)$$

Assume also that the filtered prediction error is equal to the difference between the original signal and the output of the filter:

$$\tilde{e}_p(k+1) = x_o(k+1) - \hat{x}(k+1) \quad (16)$$

$\mathbf{h}(k+1)$ is to be determined such that

$$x_o(k+1) = \mathbf{h}^T(k+1)\mathbf{x}_N(k). \quad (17)$$

Equations (15), (16) and (17) can be written as

$$\begin{aligned} \tilde{e}_p(k+1) &= [\mathbf{h}(k+1) - \mathbf{h}(k)]^T \mathbf{x}_N(k) \\ &= \Delta\mathbf{h}^T(k)\mathbf{x}_N(k). \end{aligned} \quad (18)$$

The correction vector $\Delta\mathbf{h}(k)$ is given by the first order difference approximation,

$$\Delta\mathbf{h}(k) \approx \left.\frac{\partial\mathbf{h}}{\partial\theta_1}\right|_{\hat{\theta}_1(k)} \Delta\theta_1 = \mathbf{g}(k)\Delta\hat{\theta}_1. \quad (19)$$

Substituting (19) in (18)

$$\tilde{e}_p(k+1) = \mathbf{g}^T(k)\mathbf{x}_N(k)\Delta\hat{\theta}_1 \quad (20)$$

The correction in the estimated frequency is then,

$$\Delta\hat{\theta}_1 = \frac{\tilde{e}_p(k+1)}{\mathbf{g}^T(k)\mathbf{x}_N(k)}. \quad (21)$$

The updating scheme for the estimated frequency is

$$\hat{\theta}_1(k+1) = \hat{\theta}_1(k) + \mu\frac{\tilde{e}_p(k+1)}{\mathbf{g}^T(k)\mathbf{x}_N(k)}. \quad (22)$$

where μ is a suitably chosen step size. There is a trade-off between small estimation variance (small μ, see (39)) and increased tracking ability (large μ). The updated filter coefficient vector and its gradient are

$$\mathbf{h}(k+1) = \mathbf{h}_o[\hat{\theta}_1(k+1)], \tag{23}$$

$$\mathbf{g}(k+1) = \mathbf{g}_o[\hat{\theta}_1(k+1)]. \tag{24}$$

where $\mathbf{g}_o$ is the gradient of $\mathbf{h}_o$ with respect to the frequency θ_1. It should be noted that the gradient $\mathbf{g}_o$ is efficiently calculated using finite differences from $\mathbf{h}_o$ in the specified frequency range.

Note that the denominator in (21) can become arbitrarily small since it is a linear combination of noisy sinusoids with time-varying coefficients. In such a case, the correction in the frequency cannot be solved from (21). Higher order terms may be required in the series expansion in (19) for the solution of $\Delta\theta_1$. However, in such a case, $\tilde{e}_p$ will become nonlinear in $\Delta\theta_1$. This is avoided by equating the correction $\Delta\theta_1$ to zero in such a singular case, where μ is set to zero whenever

$$\left|\mathbf{g}^T(k)\mathbf{x}_N(k)\right| < \epsilon. \tag{25}$$

Here, ϵ is a threshold for successive updates. When ϵ is too small, it may lead to instability and on the other hand, a large value of ϵ may result in decreased tracking performance. In the implementation of this scheme, polynomials of the appropriate degree are used to approximate accurately the coefficient vector $\mathbf{h}$ and its gradient $\mathbf{g}$ with respect to θ_1. Quadratic approximating polynomials are used to write

$$\begin{aligned}\mathbf{h}_{o2}(\theta_1) &= \mathbf{a} + \mathbf{b}\theta_1 + \mathbf{c}\theta_1^2 \\ \mathbf{g}_2(\theta_1) &= \boldsymbol{\alpha} + \boldsymbol{\beta}\theta_1 + \boldsymbol{\gamma}\theta_1^2\end{aligned} \tag{26}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^N$ are vectors computed to minimize the mean squared error between $\mathbf{h}_o$ (or $\mathbf{g}$) and the polynomial $\mathbf{h}_{o2}$ (or $\mathbf{g}_2$).

4. BIAS OF THE FREQUENCY ESTIMATE

The discrete-time equation for the estimated frequency $\hat{\theta}_1$ is a nonlinear stochastic equation. An approximate small-signal stability analysis can be performed around the frequency θ_1 of the original sinusoid x_o. In order to simplify the analysis, it will be assumed that there is no filtering on the prediction error. A perturbation of the estimated frequency can be defined as

$$\delta\hat{\theta}_1(k) = \hat{\theta}_1 - \theta_1. \tag{27}$$

With this definition and the above assumption, the following discrete-time equation can be written

$$\delta\hat{\theta}_1(k+1) = \delta\hat{\theta}_1(k+1) + \mu\frac{e_p(k+1)}{\mathbf{g}^T(k)\mathbf{x}_N(k)}. \tag{28}$$

Now, the prediction error in (28), using (14) is

$$e_p(k+1) = x_0(k+1) + q(k+1) - \hat{x}(k+1). \tag{29}$$

Expanding the filter coefficient vector around θ_1,

$$\begin{aligned}\mathbf{h}(k) &= \mathbf{h}_o[\hat{\theta}_1(k)] = \mathbf{h}_o[\theta_1 + \delta\hat{\theta}_1(k)] \\ &\approx \mathbf{h}_o(\theta_1) + \mathbf{g}(\theta_1)\delta\hat{\theta}_1(k)\end{aligned} \tag{30}$$

the predicted signal in (29) can then be written as

$$\begin{aligned}\hat{x}(k+1) &= \mathbf{h}^T(k)\mathbf{x}_N(k) \\ &\approx \mathbf{h}_o^T(\theta_1)\mathbf{x}_N(k) + \mathbf{g}^T(\theta_1)\mathbf{x}_N(k)\delta\hat{\theta}_1(k).\end{aligned} \tag{31}$$

Based on the measured signal model $x(k) = x_o(k) + q(k)$, the data vector in (31) can be written as

$$\mathbf{x}_N(k) = \mathbf{x}_{o,N}(k) + \mathbf{Q}(k) \tag{32}$$

where

$$\mathbf{Q}(k) = [q(k)\; q(k-1) \ldots q(k-N+1)]^T. \tag{33}$$

This leads to

$$\begin{aligned}\mathbf{h}_o^T(\theta_1)\mathbf{x}_N(k) &= \mathbf{h}_o^T(\theta_1)\mathbf{x}_{o,N}(k) + \mathbf{h}_o^T(\theta_1)\mathbf{Q}(k) \\ &= x_o(k+1) + \tilde{q}(k+1).\end{aligned} \tag{34}$$

In (34), $\tilde{q}(k+1)$ is a noise component with variance equal to that of $q(k+1)$ multiplied by the noise gain of the filter $\mathbf{h}_o(\theta_1)$ given by $G_n = \|\mathbf{h}_o\|^2$. From (31), (34) and (29),

$$e_p(k+1) = q(k+1) - \tilde{q}(k+1) - \mathbf{g}^T(\theta_1)\mathbf{x}_N(k)\delta\hat{\theta}_1(k) \tag{35}$$

is obtained. Substituting (35) in (28) and with the approximation $\mathbf{g}(k) \approx \mathbf{g}(\theta_1)$, (28) becomes

$$\delta\hat{\theta}_1(k+1) = (1-\mu)\delta\hat{\theta}_1(k) + \mu\frac{q(k+1) - \tilde{q}(k+1)}{\mathbf{g}^T(\theta_1)\mathbf{x}_N(k)}. \tag{36}$$

Taking the expectation of (36) and neglecting the random component of $\mathbf{g}^T(\theta_1)\mathbf{x}_N(k)$,

$$E\left\{\delta\hat{\theta}_1(k+1)\right\} = (1-\mu)E\left\{\delta\hat{\theta}_1(k)\right\}. \tag{37}$$

If $0 < \mu < 2$ then

$$\lim_{k\to\infty} E\left\{\delta\hat{\theta}_1(k)\right\} = 0, \tag{38}$$

which shows that the estimated frequency is unbiased. Using (36), it is also possible to obtain an approximate expression for the asymptotic variance of the frequency estimate $v(k)$ given by,

$$\lim_{k\to\infty} v(k) \approx \frac{2\mu^2\sigma_q^2(1+G_n)}{\epsilon A\theta_1(1-\alpha^P)}\sqrt{1 - \frac{\epsilon^2}{A^2}}. \tag{39}$$

where $G_n = \|\mathbf{h}_o(\theta_1)\|^2$, $\alpha = (1-\mu)^2$, $P = \pi/\theta_1$ and A is the amplitude of $\mathbf{g}^T(\theta_1)\mathbf{x}_N$.

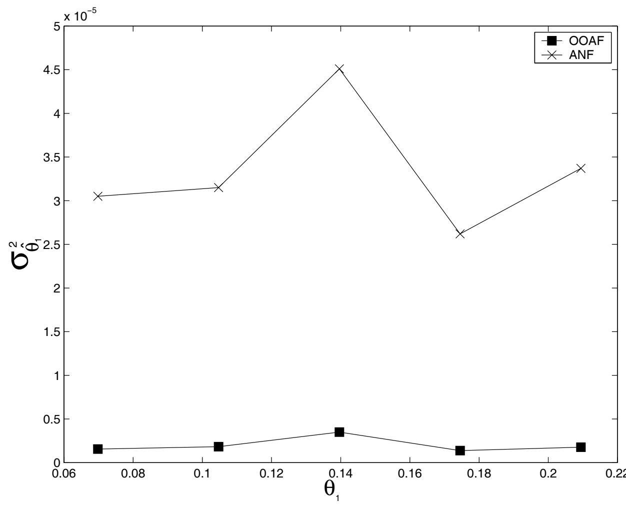

Fig. 1. Variance of the estimated frequency versus the signal frequency θ_1 in colored noise. $\mu = 0.01$, $\epsilon = 5$, $\sigma_q^2 = 0.16$.

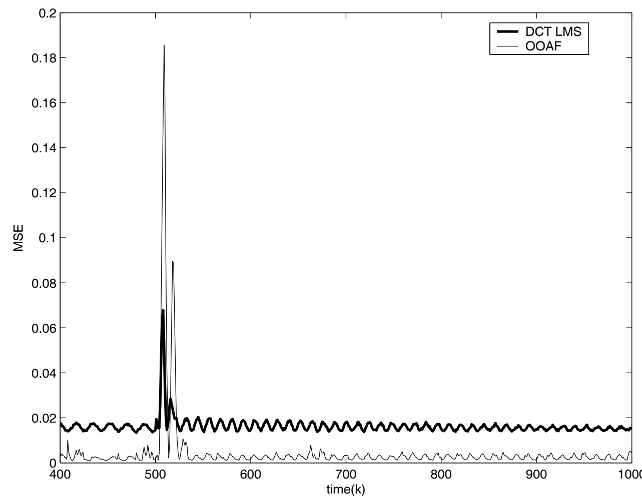

Fig. 2. The waveform estimation performance in colored noise. $\theta_1 = \pi/18$, $\mu = 0.25$, $\epsilon = 8$, $\sigma_q^2 = 0.025$, ES=11.

5. COMPUTER SIMULATIONS

The offline filter is designed in the frequency range [$\pi/45$, $\pi/15$]. The offline optimized filter coefficient vector $\mathbf{h_o}$ and its gradient $\mathbf{g_o}$ are computed. The adaptive filter of length $N = 20$ is applied to the observed noisy signal $x(k)$ to obtain its prediction $\hat{x}(k+1)$ with an error of $e_p(k+1)$. It should be noted that it is possible to vary N depending on the requirements set by some applications. Increasing N leads to better noise suppression performance but it also increases complexity. In Fig. 1 a performance comparison is made in correlated Gaussian noise, obtained by first band-pass filtering AWGN. There is an approximately 15-fold decrease in variance relative to the Adaptive Notch Filter (ANF) over the same range of frequencies [2]. Figure 2 shows waveform estimation performance of OOAF in colored noise where the eigenvalue spread (ES) is 11. The variance is shown where the frequency estimate converges from an initial value of 0.087 rad. to the original signal frequency of 0.1397 rad. The performance is compared with DCT-LMS which reduces ES and enables LMS to converge to the input waveform. The stepsize μ of OOAF is adjusted to have the same tracking speed as the DCT-LMS. It is observed that the ensemble-averaged mse associated with the proposed OOAF is consistently smaller than that of DCT-LMS in the steady state. The computational complexity of ANF is lower than the proposed OOAF algorithm which has a complexity comparable to the conventional LMS algorithm for waveform estimation. The number of multiplications required for OOAF is approximately $8N$ and the number of additions is $5N$ where N is the filter length. For waveform estimation in colored noise, DCT-LMS requires much higher computational complexity in order to evaluate the DCT transform coefficients at each iteration.

6. CONCLUSIONS

A novel approach to the estimation of a single-tone sinusoid and its frequency is presented. The frequency estimate is shown to be unbiased and an approximate analytical expression has been derived for its asymptotic variance. The method results in rapid convergence to the true frequency and consistently tracks the sinusoidal waveform. In correlated Gaussian noise, over the chosen range, the method is superior to ANF in estimating frequency. Waveform estimation is also shown to have a higher performance compared to DCT-LMS with a much lower computational complexity.

7. REFERENCES

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